Rutgers University: Algebra Written Qualifying Exam

January 2006: Day 1 Problem 5 Solution

Exercise. Prove that, for any group G, the set of all automorphisms of the form $\phi(G) = xgx^{-1}$, $x \in G$ is a normal subgroup of the group of all automorphisms of G.

Solution.

Prove subgroup:

Let $S = \{\phi \in Aut(G) : \phi(g) = xgx^{-1}, x \in G\}$

Obviously $S \subseteq Aut(G)$.

$$
\begin{aligned}
\text{If } \phi_1, \phi_2 \in S, \quad \text{then} \quad \phi_1(g) &= x_1 g x_1^{-1} \quad \text{and} \quad \phi_2(g) = x_2 g x_2^{-1} \quad \text{for some } x_1, x_2 \in G \\
\phi_2 \circ \phi_1(g) &= \phi_2(x_1 g x_1^{-1}) \\
&= x_2 x_1 g x_1^{-1} x_2^{-1} \\
&= (x_2 x_1) g (x_2 x_1)^{-1} \quad \text{and} \quad x_2 x_1 \in G \\
\implies \quad \phi_2 \circ \phi_1 &\in S \\
\text{Also,} \quad \phi_1^{-1}(g) &= x_1^{-1} g x_1 \\
&= (x_1^{-1}) g (x_1^{-1})^{-1} \quad \text{and} \quad x_1^{-1} \in G \\
\implies \quad \phi_1^{-1} &\in S
\end{aligned}
$$

Thus, S is a subgroup of $Aut(G)$.

Prove Normal: S is normal in $Aut(g)$ if for any $\psi \in Aut(G)$, $\phi_1 \in S$, there exists $\phi_2 \in S$ such that

$$\psi \circ \phi_1(g) = \phi_2 \circ \psi(g)$$

$$
\begin{aligned}
\psi \circ \phi_1(g) &= \psi(x_1 g x_1^{-1}) \\
&=_1) psi(g)[\psi(x_1)]^{-1} && \text{since } \psi \text{ is a homomorphism} \\
&= x_3 \psi(g) x_2^{-1} && \text{for } x_2 = \psi(x_1) \in G \text{ since } \psi \text{ an automorphism} \\
&= \phi_2 \circ \psi(g) && \text{where } \phi_2 = x_1 = x_2 g x_2^{-1} \text{ and } x_2 \in G \\
\implies \phi_2 &\in S
\end{aligned}
$$

Thus, S is a normal subgroup of $Aut(G)$